Diploma Programme
Programme du diplôme
Programa del Diploma

Markscheme

November 2020

Further mathematics

Higher level

Paper 2

21 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a valid **Method**; working must be seen.

(M) Marks awarded for **Method**; may be implied by **correct** subsequent working.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

(A) Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.

R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Mark according to RM™ Assessor instructions. In particular, please note the following:

- Marks must be recorded using the annotation stamps. Please check that you are entering marks for the right question.
- If a part is **completely correct**, (and gains all the "must be seen" marks), use the ticks with numbers to stamp full marks.
- If a part is completely wrong, stamp ***A0*** by the final answer.
- If a part gains anything else, it **must** be recorded using **all** the annotations.
- All the marks will be added and recorded by RM™ Assessor.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *eg* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*eg* substitution into a formula) and ***A1*** for using the **correct** values.
- Where the markscheme specifies ***(M2)***, ***N3***, *etc.*, do **not** split the marks.

- Once a correct answer to a question or part-question is seen, ignore further correct working. However, if further working indicates a lack of mathematical understanding do not award the final ***A1***. An exception to this may be in numerical answers, where a correct exact value is followed by an incorrect decimal. However, if the incorrect decimal is carried through to a subsequent part, and correct ***FT*** working shown, award ***FT*** marks as appropriate but do not award the final ***A1*** in that part.

  Examples

	Correct answer seen	**Further working seen**	**Action**
1.	$8\sqrt{2}$	$5.65685...$ *(incorrect decimal value)*	Award the final ***A1*** *(ignore the further working)*
2.	$\frac{1}{4}\sin 4x$	$\sin x$	Do not award the final ***A1***
3.	$\log a - \log b$	$\log(a-b)$	Do not award the final ***A1***

3 *N* marks

*Award **N** marks for **correct** answers where there is **no** working.*

- Do **not** award a mixture of ***N*** and other marks.
- There may be fewer ***N*** marks available than the total of ***M***, ***A*** and ***R*** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

4 Implied marks

*Implied marks appear in **brackets eg (M1)**, and can only be awarded if **correct** work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

5 Follow through marks

*Follow through (**FT**) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s). To award **FT** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part.*

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks.
- If the error leads to an inappropriate value (*eg* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).
- Within a question part, once an error is made, no further **dependent *A*** marks can be awarded, but ***M*** marks may be awarded if appropriate.
- Exceptions to this rule will be explicitly noted on the markscheme.

6 Misread

*If a candidate incorrectly copies information from the question, this is a misread (**MR**). A candidate should be penalized only once for a particular misread. Use the **MR** stamp to indicate that this has been a misread. Then deduct the first of the marks to be awarded, even if this is an **M** mark, but award all others so that the candidate only loses **[1 mark]**.*

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*eg* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).

7 Discretionary marks *(d)*

*An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. In such cases the annotation DM should be used and a brief **note** written next to the mark explaining this decision.*

8 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for part-questions are indicated by **EITHER . . . OR**.
- Where possible, alignment will also be used to assist examiners in identifying where these alternatives start and finish.

9 Alternative forms

*Unless the question specifies otherwise, **accept** equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

Example: for differentiating $f(x) = 2\sin(5x-3)$, the markscheme gives

$$f'(x) = \left(2\cos(5x-3)\right)5 \quad \left(=10\cos(5x-3)\right) \qquad \textbf{\textit{A1}}$$

Award ***A1*** for $\left(2\cos(5x-3)\right)5$, even if $10\cos(5x-3)$ is not seen.

10 Accuracy of Answers

*Candidates should **NO LONGER** be penalized for an accuracy error **(AP)**.*

*If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy. When this is not specified in the question, all numerical answers should be given exactly or correct to three significant figures. Please check work carefully for **FT**.*

11 Crossed out work

If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.

12 Calculators

A GDC is required, but calculators with symbolic manipulation features (for example, TI-89) are not allowed.

Calculator notation
The Mathematics HL guide says:
Students must always use correct mathematical notation, not calculator notation.
Do **not** accept final answers written using calculator notation. However, do not penalize the use of calculator notation in the working.

13 More than one solution

Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.

1. (a) $411 = 1 \times 339 + 72$ ***M1***
$339 = 4 \times 72 + 51$ ***A1***
$72 = 1 \times 51 + 21$ ***A1***
$51 = 2 \times 21 + 9$ ***M1***
$21 = 2 \times 9 + 3$ ***A1***
$(9 = 3 \times 3 + 0)$
the GCD is 3 ***AG***

[5 marks]

(b) (i) reversing the process,
$3 = 21 - 2 \times 9$ ***(M1)***
$= 21 - 2 \times (51 - 2 \times 21) = 5 \times 21 - 2 \times 51$ ***(A1)***
$= 5 \times (72 - 51) - 2 \times 51 = 5 \times 72 - 7 \times 51$
$= 5 \times 72 - 7 \times (339 - 4 \times 72) = 33 \times 72 - 7 \times 339$ ***(A1)***
$= 33 \times (411 - 339) - 7 \times 339$
$= 33 \times 411 - 40 \times 339$ ***(A1)***
therefore $x_0 = 33$, $y_0 = 40$ is a solution to the equation ***(A1)***
the general solution is $x = 33 + 113N, y = 40 + 137N$ ***A1***

Note: Accept the tracking of linear combinations when applying the Euclidean algorithm (could be displayed in in part (a)).

Note: Award ***A1FT*** for a candidate's $x = x_0 + 113N$ and $y = y_0 + 137N$.

(ii) dividing by 3 it follows from the above that
$137 \times 33 = 1 + 40 \times 113 \equiv 1 \pmod{113}$ ***(M1)***
thus $x = 33$ is a solution to the congruence ***(A1)***
the general solution is $x = 33 + 113N$ $\left(x \equiv 33 \pmod{113}\right)$ ***A1***

[9 marks]

Total [14 marks]

2. (a) **EITHER**

forms $\boldsymbol{AA}^{-1}$ ***(M1)***

$$\begin{bmatrix}1 & a & b\\0 & 1 & c\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & -a & ac-b\\0 & 1 & -c\\0 & 0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}1 & -a+a & ac-b-ac+b\\0 & 1 & -c+c\\0 & 0 & 1\end{bmatrix}$$ ***A1***

OR

forms $\boldsymbol{A}^{-1}\boldsymbol{A}$ ***(M1)***

$$\begin{bmatrix}1 & -a & ac-b\\0 & 1 & -c\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & a & b\\0 & 1 & c\\0 & 0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}1 & -a+a & ac-b-ac+b\\0 & 1 & -c+c\\0 & 0 & 1\end{bmatrix}$$ ***A1***

THEN

$$=\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$ which proves the result ***AG***

> **Note:** Award ***M1*** for forming $\boldsymbol{AA}^{-1}=\boldsymbol{I}$ or $\boldsymbol{A}^{-1}\boldsymbol{A}=\boldsymbol{I}$ where $\boldsymbol{A}^{-1}=\begin{bmatrix}1 & x & y\\0 & 1 & z\\0 & 0 & 1\end{bmatrix}$
> and ***A1*** for clearly determining that $x=-a$, $y=ac-b$ and $z=-c$.

[2 marks]

(b) (i) closure:

consider $\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & d & e \\ 0 & 1 & f \\ 0 & 0 & 1 \end{bmatrix}$ both of which belong to S ***M1***

$$\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & d & e \\ 0 & 1 & f \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & d+a & e+af+b \\ 0 & 1 & f+c \\ 0 & 0 & 1 \end{bmatrix}$$ ***A1***

which belongs to S, therefore closed

identity:
putting $a, b, c = 0$, the identity matrix is seen to belong to S ***A1***

inverse:
it has been shown in (a) that a matrix belonging to S has an inverse in S ***A1***

associativity:
this follows since matrix multiplication is associative ***A1***
the four group axioms are satisfied therefore $\{S, *\}$ is a group ***AG***

(ii) consider $\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & d & e \\ 0 & 1 & f \\ 0 & 0 & 1 \end{bmatrix}$ both of which belong to S ***M1***

$$\begin{bmatrix} 1 & d & e \\ 0 & 1 & f \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a+d & b+dc+e \\ 0 & 1 & c+f \\ 0 & 0 & 1 \end{bmatrix}$$ ***A1***

this is different from the reverse product found above so $\{S, *\}$
is not Abelian ***A1***

Note: Condone the correct use of a specific counterexample to demonstrate that $\{S, *\}$ is not Abelian.

(iii) **EITHER**

$$\begin{bmatrix}1 & a & b\\ 0 & 1 & c\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & a & b\\ 0 & 1 & c\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}1 & 2a & 2b+ac\\ 0 & 1 & 2c\\ 0 & 0 & 1\end{bmatrix}$$ ***M1***

$$=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\Rightarrow a=b=c=0$$ ***A1***

OR

compares $\boldsymbol{A}$ and $\boldsymbol{A}^{-1}$ ***M1***

determines from this comparison that $a=b=c=0$ ***A1***

THEN

the only self-inverse element is therefore the identity matrix ***A1***

[11 marks]

(c) yes because the identity, inverse and the product would belong to T ***A1***

[1 mark]

Total [14 marks]

3. (a) PQRSTUVWP or PWVUTSRQP ***A1***

[1 mark]

(b) (i) G is not bipartite because it contains triangles (odd cycles) ***A1***

Note: Accept an adjacency argument showing that a particular vertex (for example, W) cannot belong to two disjoint vertex sets. The feature in this instance is the particular vertex.

(ii) G does not have an Eulerian trail because it has more than two vertices (four vertices, Q,V,R and S) of odd degree ***R1***

(iii) remove an edge joining two vertices of odd degree: Q,V,R and S ***(M1)***

H would now have an Eulerian trail

three possible edges are:

QR or RS or RV ***A2***

Note: Award ***A1*** for two correct possible edges.

[5 marks]

(c) (i) **EITHER**

step	Vertices labelled	Working values	
1	P	P(0), W-9, Q-5	***M1***
2	P, Q	P(0), Q(5), W-8, R-14	***A1***
3	P, Q, W	P(0), Q(5), W(8),R-11, V-10	***A1***
4	P, Q, W, V	P(0), Q(5), W(8), V(10), R-11, U-22	***A1***
5	P, Q, W, V, R	P(0), Q(5), W(8), V(10), R(11), U-16, S-23	
6	P, Q, W, V, R, U	P(0), Q(5), W(8), V(10), R(11), U(16), S-18, T-24	***A1***
7	P, Q, W, V, R, U, S	P(0), Q(5), W(8), V(10), R(11), U(16), S(18), T-22	
8	P, Q, W, V, R, U, S, T	P(0), Q(5), W(8), V(10), R(11), U(16), S(18), T(22)	***A1***

OR

	P	Q	R	S	W	V	U	T	
P	$\boxed{0}$	5			9				***M1***
Q		$\boxed{5}$	14		8				***A1***
R			11		$\boxed{8}$	10			***A1***
S						$\boxed{10}$	22		
W			$\boxed{11}$	23			16		***A1***
V				18			$\boxed{16}$	24	
U				$\boxed{18}$				22	***A1***
T								$\boxed{22}$	

back tracking line shown ***(A1)***

THEN

the minimum weight path is PQWRUST ***A1***

(ii) with weight 22 ***A1***

[8 marks]

Total [14 marks]

4. (a) **METHOD 1**

attempts to find the gradient of one of $[\mathrm{OS}]$ or $[\mathrm{OT}]$ ***(M1)***

$m_{[\mathrm{OS}]} = \frac{2as}{as^2} = \frac{2}{s}$ and $m_{[\mathrm{OT}]} = \frac{2at}{at^2} = \frac{2}{t}$ ***A1***

(condition for perpendicularity is) $\frac{2}{s} \times \frac{2}{t} = -1$ ***A1***

$\Rightarrow st = -4$ ***AG***

METHOD 2

forms $\begin{pmatrix} as^2 \\ 2as \end{pmatrix} \cdot \begin{pmatrix} at^2 \\ 2at \end{pmatrix}$ ***(M1)***

(condition for perpendicularity is) $a^2s^2t^2 + 4a^2st = 0$ ***A1***

$a^2st(st+4) = 0$ and $a^2st \neq 0$ ***A1***

$\Rightarrow st = -4$ ***AG***

[3 marks]

Note: In parts (b), (c) and (d), accept solutions which replace t by $-\frac{4}{s}$ or s by $-\frac{4}{t}$ at any stage.

(b) attempts to find the gradient of (ST) ***(M1)***

$m_{(\mathrm{ST})} = \frac{2as - 2at}{as^2 - at^2}$

$= \frac{2}{s+t}$ ***A1***

Note: The ***A1*** for simplification can be awarded at any stage.

the equation of (ST) is $y - 2as = \frac{2}{s+t}(x - as^2)$ ***A1***

(ST) meets the x- axis where $y = 0$ ***M1***

$x - as^2 = -as(s+t) \Rightarrow x = -ast$ ***A1***

$x = 4a$ ***A1***

Note: Award as above for $y - 2at = \frac{2}{s+t}(x - at^2)$.

which is independent of s, t ***AG***

[6 marks]

continued…

(c) attempts to find the gradient of the tangent at S or T **M1**

for example, at S, $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}s} \div \frac{\mathrm{d}x}{\mathrm{d}s} = \frac{1}{s}$ $\left(= \frac{2a}{y}\right)$

the equation of the tangent at S is

$y - 2as = \frac{1}{s}\left(x - as^2\right)$ $\left(y = \frac{1}{s}x + as\right)$ **A1**

the equation of the tangent at T is

$y - 2at = \frac{1}{t}\left(x - at^2\right)$ $\left(y = \frac{1}{t}x + at\right)$ **A1**

EITHER

these tangents intersect where

$2as + \frac{1}{s}\left(x - as^2\right) = 2at + \frac{1}{t}\left(x - at^2\right)$ **M1**

$2as^2t + tx - as^2t = 2ast^2 + sx - ast^2$ **A1**

$x(s-t) = ast(s-t) \Rightarrow x = ast$ **A1**

OR

these tangents intersect where

$\frac{1}{s}x + as = \frac{1}{t}x + at$ **M1**

$x\left(\frac{1}{s} - \frac{1}{t}\right) = a\left(t - s\right)$ $\left(\left(\frac{t-s}{st}\right)x = a\left(t-s\right)\right)$ **A1**

$x(s-t) = ast(s-t) \Rightarrow x = ast$ **A1**

THEN

$x = ast \Rightarrow x = -4a$

$x + 4a = 0$ **AG**

[6 marks]

(d) (i) the coordinates of the midpoint of $[\mathrm{ST}]$ are

$x = \frac{1}{2}\left(as^2 + at^2\right)$ $\left(= \frac{1}{2}a\left(s^2 + t^2\right)\right)$, $y = \frac{1}{2}\left(2as + 2at\right)$ $\left(= a(s+t)\right)$ **A1**

attempts to form an expression for y^2 in terms of a, s, t **M1**

$y^2 = a^2\left(s^2 + t^2 + 2st\right)$ $\left(= a^2(s^2 + t^2) - 8a^2\right)$ **A1**

uses $2x = a\left(s^2 + t^2\right)$ (or equivalent) to eliminate s, t **M1**

$2ax = y^2 + 8a^2$ **A1**

which is the equation of the locus showing it to be a parabola **AG**

(ii) rewrites the equation in the form $y^2 = 2a(x-4a)$ ***M1***

Note: Could be seen in part (d) (i).

vertex: $(4a, 0)$ ***A1***

focus: $\left(\frac{9}{2}a, 0\right)$ ***A1***

[8 marks]

Total [23 marks]

5. (a) **METHOD 1**

attempts to substitute $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \ldots$ into

$\cos x = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \ldots$ ***M1***

$\cos(\ln(1+x)) = 1 - \frac{1}{2}\left(x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \ldots\right)^2 + \frac{1}{24}(x + \ldots)^4 + \ldots$ ***A1***

attempts to expand the RHS up to and including the x^4 term ***M1***

$= 1 - \frac{1}{2}\left(x^2 - x^3 + \frac{1}{4}x^4 + \frac{2}{3}x^4 \ldots\right) + \frac{1}{24}x^4 + \ldots$ ***A1***

$= 1 - \frac{1}{2}x^2 + \frac{1}{2}x^3 - \frac{5}{12}x^4 + \ldots$ ***AG***

METHOD 2

attempts to substitute $\ln(1+x)$ into $\cos x = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \ldots$ ***M1***

$\cos(\ln(1+x)) = 1 - \frac{1}{2}(\ln(1+x))^2 + \frac{1}{24}(\ln(1+x))^4 - \ldots$

attempts to find the Maclaurin series for $(\ln(1+x))^2$ up to and including the x^4 term

M1

$(\ln(1+x))^2 = x^2 - x^3 + \frac{11}{12}x^4 - \ldots$ ***A1***

$(\ln(1+x))^2 = x^4 - \ldots$

$= 1 - \frac{1}{2}\left(x^2 - x^3 + \frac{11}{12}x^4 + \ldots\right) + \frac{1}{24}x^4 + \ldots$ ***A1***

$= 1 - \frac{1}{2}x^2 + \frac{1}{2}x^3 - \frac{5}{12}x^4 + \ldots$ ***AG***

[4 marks]

(b) $-\sin\left(\ln(1+x)\right)\times\frac{1}{1+x}=-x+\frac{3}{2}x^2-\frac{5}{3}x^3+\ldots$ ***A1A1***

$\sin\left(\ln(1+x)\right)=-(1+x)\left(-x+\frac{3}{2}x^2-\frac{5}{3}x^3+\ldots\right)$

attempts to expand the RHS up to and including the x^3 term ***M1***

$=x-\frac{3}{2}x^2+\frac{5}{3}x^3+x^2-\frac{3}{2}x^3+\ldots$ ***A1***

$=x-\frac{1}{2}x^2+\frac{1}{6}x^3+\ldots$ ***AG***

[4 marks]

(c) **METHOD 1**

let $\tan\left(\ln(1+x)\right)=a_0+a_1x+a_2x^2+a_3x^3+\ldots$

uses $\sin\left(\ln(1+x)\right)=\cos\left(\ln(1+x)\right)\times\tan\left(\ln(1+x)\right)$ to form ***M1***

$x-\frac{1}{2}x^2+\frac{1}{6}x^3+\ldots=\left(1-\frac{1}{2}x^2+\frac{1}{2}x^3+\ldots\right)(a_0+a_1x+a_2x^2+a_3x^3+\ldots)$ ***A1***

$=a_0+a_1x+\left(a_2-\frac{1}{2}a_0\right)x^2+\left(a_3-\frac{1}{2}a_1+\frac{1}{2}a_0\right)x^3+\ldots$ ***(A1)***

attempts to equate coefficients,

$a_0=0,\ a_1=1,\ a_2-\frac{1}{2}a_0=-\frac{1}{2},\ a_3-\frac{1}{2}a_1+\frac{1}{2}a_0=\frac{1}{6}$ ***M1***

$a_0=0,\ a_1=1,\ a_2=-\frac{1}{2},\ a_3=\frac{2}{3}$ ***A1***

so $\tan\left(\ln(1+x)\right)=x-\frac{1}{2}x^2+\frac{2}{3}x^3+\ldots$

METHOD 2

uses $\tan\left(\ln(1+x)\right)=\frac{\sin\left(\ln(1+x)\right)}{\cos\left(\ln(1+x)\right)}$ to form ***M1***

$=\left(x-\frac{1}{2}x^2+\frac{1}{6}x^3+\ldots\right)\left(1-\frac{1}{2}x^2+\frac{1}{2}x^3+\ldots\right)^{-1}$ ***A1***

$\left(1-\frac{1}{2}x^2+\frac{1}{2}x^3+\ldots\right)^{-1}=1+\frac{1}{2}x^2-\frac{1}{2}x^3+\ldots$ ***(A1)***

attempts to expand the RHS up to and including the x^3 term ***M1***

$=\left(x-\frac{1}{2}x^2+\frac{1}{6}x^3+\ldots\right)\left(1+\frac{1}{2}x^2-\frac{1}{2}x^3+\ldots\right)$

$=x+\frac{1}{2}x^3-\frac{1}{2}x^2+\frac{1}{6}x^3+\ldots$

$=x-\frac{1}{2}x^2+\frac{2}{3}x^3+\ldots$ ***A1***

Note: Accept use of long division.

[5 marks]

Total [13 marks]

6. (a) (i) reflexive:

since p divides $a^2 - a^2 = 0$, it follows that aRa ***A1***
therefore R is reflexive

symmetric:
let aRb so that p divides $a^2 - b^2$ ***M1***
it follows that p divides $b^2 - a^2$ so that bRa ***A1***
therefore R is symmetric

transitive:
let aRb and bRc so that $a^2 - b^2$ and $b^2 - c^2$ are divisible by p ***M1***
it follows that $a^2 - b^2 + b^2 - c^2 = a^2 - c^2$ is divisible by p so that aRc ***M1A1***
therefore R is transitive
hence R is an equivalence relation ***AG***

(ii) $aR1$ if $a^2 - 1$ is divisible by p ***(M1)***
it follows that $(a+1)(a-1)$ is divisible by p ***A1***
(since p is prime and $p > 2$), p divides $(a+1)$ or p divides $(a-1)$ ***R1***
the smallest positive integers related to 1 are found
by putting $a+1 = p$ or $a-1 = p$ ***(M1)***
$p-1$ and $p+1$ (are the two smallest positive integers related to 1) ***A1***

[11 marks]

(b) (i) $|x-2| = k^3 - 1 \Rightarrow x = \pm\left(k^3 - 1\right) + 2$ ***(M1)***

the required integers are $9(k=2)$ and $28(k=3)$ ***A1A1***

(ii) S is reflexive since $1 + |x - x| = 1^3$ ***A1***

S is symmetric since $1 + |x - y| = 1 + |y - x|$ ***A1***

for example, considers $9S2$ and $2S28$ ***M1***

$9S28$ is not true since $1 + |28 - 9|$ is not a perfect cube so S is not transitive
R1

so only two of the three requirements for S are satisfied ***AG***

[7 marks]

Total [18 marks]

7. (a) attempts to find $\mathrm{E}(X)$ ***M1***

$\mathrm{E}(X) = \int_0^a x \times \frac{4x^3}{a^4} \mathrm{d}x$

$= \left[\frac{4x^5}{5a^4}\right]_0^a$ ***A1***

$= \frac{4}{5}a$ ***A1***

attempts to show that $\mathrm{E}(\hat{a}) = a$ where $\hat{a} = \frac{5}{4}\bar{X}$

$\mathrm{E}(\hat{a}) = \frac{5}{4}\mathrm{E}(\bar{X})$ ***M1***

$= \frac{5}{4} \times \frac{4}{5}a = a$ ***A1***

therefore $\hat{a}$ is an unbiased estimator for a ***AG***

[5 marks]

(b) attempts to find $\mathrm{E}(X^2)$ ***M1***

$\mathrm{E}(X^2) = \int_0^a x^2 \times \frac{4x^3}{a^4} \mathrm{d}x$

$= \left[\frac{4x^6}{6a^4}\right]_0^a$ ***A1***

$= \frac{2}{3}a^2$ ***A1***

attempts to use $\mathrm{Var}(X) = \mathrm{E}(X^2) - (\mathrm{E}(X))^2$ ***M1***

$\mathrm{Var}(X) = \frac{2}{3}a^2 - \left(\frac{4}{5}a\right)^2$

$= \frac{2}{75}a^2$ ***A1***

> **Note:** Award ***M1M1A1A1A1*** for correct use of $\mathrm{Var}(X) = \mathrm{E}(X - \mathrm{E}(X))^2$ leading to $\mathrm{Var}(X) = \frac{2}{75}a^2$.

attempts to find $\mathrm{Var}(\hat{a})$ ***M1***

$\mathrm{Var}(\hat{a}) = \frac{25}{16} \times \mathrm{Var}(\bar{X})$

$= \frac{25}{16} \times \frac{\mathrm{Var}(X)}{n}$ ***A1***

$= \frac{25}{16} \times \frac{2}{75} \times \frac{a^2}{n}$

$= \frac{a^2}{24n}$ ***AG***

[7 marks]

Question 7 continued

(c) (i) using the CLT for $\overline{X}$

$\hat{a} = \frac{5}{4}\overline{X}$ approximately follows $\mathrm{N}(a, \frac{a^2}{24n})$ ***(M1)***

$\approx \mathrm{N}\left(15, \frac{3}{16}\right)$ ***(A1)***

the 95% interval is $15 \pm 1.96...\sqrt{0.1875}$ ***(M1)***

giving $(14.2, 15.8)$ ***A1***

Note: Accept all answers that round to the correct 3sf answer.

(ii) no ***A1***

because the term confidence interval is used to describe an interval containing a constant parameter, here it contains a random variable ***R1***

Note: Do not award ***A1R0***.

[6 marks]

Total [18 marks]

8. (a) attempts to find $\det \boldsymbol{A}$ ***M1***

$\det \boldsymbol{A} = 4\lambda - 1 + \lambda\left(2 - \lambda^2\right) - 2(\lambda - 8) \ \left(= -\lambda^3 + 4\lambda + 15\right)$ ***A1***

recognises that $\boldsymbol{A}$ is singular when $\det \boldsymbol{A} = 0$ ***M1***

solves to obtain $\lambda = 3, (-1.5 \pm 1.66\mathrm{i})$ and

demonstrates that the only real root is 3 ***R1***

for which $\boldsymbol{A}$ is singular ***AG***

[4 marks]

(b) (i) attempts row reduction ***M1***

for example, $\mathrm{R}_2 - 3\mathrm{R}_1$ and $\mathrm{R}_3 - 2\mathrm{R}_1$

$$\begin{bmatrix} 1 & 3 & -2 \\ 0 & -5 & 7 \\ 0 & -5 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \mu - 2 \end{bmatrix}$$ ***A1***

for consistency $\mu - 2 = 1$

Note: Award ***M1*** for stating $\mathrm{R}_2 = \mathrm{R}_1 + \mathrm{R}_3$ (or equivalent) and ***A1*** for stating $\mu + 1 = 4$.

$\Rightarrow \mu = 3$ ***A1***

(ii) attempts to solve by putting $z = \alpha$, for example, ***M1***

$y = \frac{7\alpha - 1}{5}, \ x = \frac{8 - 11\alpha}{5}$ ***A1A1***

[6 marks]

continued…

Question 8 continued

(c) (i) $\begin{bmatrix} 1 & -1 & -2 \\ -1 & 4 & 1 \\ 2 & 1 & -1 \end{bmatrix}^{-1}$

$= \frac{1}{12}\begin{bmatrix} -5 & -3 & 7 \\ 1 & 3 & 1 \\ -9 & -3 & 3 \end{bmatrix}$ ***A2***

(ii) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{12}\begin{bmatrix} -5 & -3 & 7 \\ 1 & 3 & 1 \\ -9 & -3 & 3 \end{bmatrix}\begin{bmatrix} a \\ 5 \\ 1 \end{bmatrix}$ ***(A1)***

attempts to find the RHS matrix product ***M1***

$= \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{12}\begin{bmatrix} -5a-8 \\ a+16 \\ (-9a-12) \end{bmatrix}$ ***A1***

$y = x^2 \Rightarrow \frac{a+16}{12} = \frac{(-5a-8)^2}{144}$ ***M1***

$\Rightarrow 25a^2 + 68a - 128 = 0$ ***(A1)***

$a = -4$ ***A1***

$x = y = 1$ ***A1***

[9 marks]

Total [19 marks]

9. (a) puts $y = vx$ so that $\frac{\mathrm{d}y}{\mathrm{d}x} = v + x\frac{\mathrm{d}v}{\mathrm{d}x}$ ***M1***

$v + x\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{vx - x}{vx + x}$ $\left(= \frac{v-1}{v+1}\right)$ ***A1***

attempts to express $x\frac{\mathrm{d}v}{\mathrm{d}x}$ as a single rational fraction in v

$x\frac{\mathrm{d}v}{\mathrm{d}x} = -\frac{v^2+1}{v+1}$ ***M1***

attempts to separate variables ***M1***

$\int \frac{v+1}{v^2+1}\mathrm{d}v = -\int \frac{1}{x}\mathrm{d}x$

$\frac{1}{2}\ln(v^2+1) + \arctan v = -\ln x\left(+C\right)$ ***A1A1***

substitutes $y = 2, x = 1$ and attempts to find the value of C ***M1***

$C = \frac{1}{2}\ln 5 + \arctan 2$ ***A1***

the solution is

$\frac{1}{2}\ln\left(\frac{y^2}{x^2}+1\right) + \arctan\left(\frac{y}{x}\right) + \ln x - \frac{1}{2}\ln 5 - \arctan 2 = 0$ ***A1***

[9 marks]

(b) at a maximum, $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ ***M1***

attempts to substitute $x = y$ into their solution ***M1***

$\frac{1}{2}\ln 2 + \arctan 1 + \ln x = \frac{1}{2}\ln 5 + \arctan 2$

attempts to solve for x, y ***(M1)***

$\left(2.18, 2.18\right)$ $\left(\frac{\sqrt{10}}{2}\mathrm{e}^{\arctan 2 - \frac{\pi}{4}}, \frac{\sqrt{10}}{2}\mathrm{e}^{\arctan 2 - \frac{\pi}{4}}\right)$ ***A1***

Note: Accept all answers that round to the correct 2sf answer.
Accept $x = 2.18, y = 2.18$.

[4 marks]

(c) **METHOD 1**

attempts (quotient rule) implicit differentiation ***M1***

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{\left(\frac{\mathrm{d}y}{\mathrm{d}x} - 1\right)(y+x) - (y-x)\left(\frac{\mathrm{d}y}{\mathrm{d}x} + 1\right)}{(y+x)^2}$$

correctly substitutes $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y-x}{y+x}$ into $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}$

$$= \frac{\left(\frac{y-x}{y+x} - 1\right)(y+x) - (y-x)\left(\frac{y-x}{y+x} + 1\right)}{(y+x)^2}$$ ***A1***

$$= -\frac{2(x^2+y^2)}{(y+x)^3}$$ ***A1***

this expression can never be zero therefore no points of inflexion ***R1***

METHOD 2

attempts implicit differentiation on $(y+x)\frac{\mathrm{d}y}{\mathrm{d}x} = y - x$ ***M1***

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x} + 1\right)\frac{\mathrm{d}y}{\mathrm{d}x} + (y+x)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{\mathrm{d}y}{\mathrm{d}x} - 1$$ ***A1***

$$(y+x)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{\mathrm{d}y}{\mathrm{d}x} - 1 - \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 - \frac{\mathrm{d}y}{\mathrm{d}x}$$

$$= -1 - \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2$$ ***A1***

$-1 - \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 < 0$ and $x + y > 0$, $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} \neq 0$ therefore no points of inflexion ***R1***

> **Note:** Accept putting $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 0$ and obtaining contradiction.

[4 marks]

Total [17 marks]